May 1997

On The Planar Yang-Mills Theory

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Abstract

The planar Yang-Mills theory in three spatial dimensions is examined in a particular representation which explicitly embodies factorization. The effective planar Yang-Mills theory Hamiltonian is constructed in this representation.

The planar Yang-Mills theory [1] (or the $N = \infty$ limit of the $SU(N)$ Yang-Mills theory) is characterized by a remarkable property of factorization, that is, $\langle XY \rangle = \langle X \rangle \langle Y \rangle$, where X and Y are some gauge-invariant observables. This statement implies "classicality" of the planar Yang-Mills theory. In this note I would like to propose a particular heuristic representation of the planar Yang-Mills theory in three spatial dimensions which explicitly embodies factorization. (The heuristic construction presented below is in some sense an extrapolation of the well-known results of [2].) I want to emphasize that most of the formulae stated below should be understood, for the time being, as mere formal expressions. The very important question of renormalization (after suitable regularization) is not considered at all. In what follows I adopt the Hamiltonian framework in three spatial dimensions. (The Planck constant is set to one.)

I would like to address the following two questions: "What is a natural representation of the $SU(\infty)$ gauge connections $A_i(\vec{x})$ such that the gauge structure of the $SU(\infty)$ Yang-Mills theory is fully preserved?", and "Given such a representation of $A_i(\vec{x})$, how does the ground state of the planar Yang-Mills theory look like, and furthermore, what is the form of the effective planar Yang-Mills Hamiltonian?".

In the $N = \infty$ limit the gauge connections $A_i(\vec{x})$ are $\infty \times \infty$ matrices at each space point. In order to make sense of this basic fact it is absolutely essential to come up with a suitable representation for $A_i(\vec{x})$ so that the matrix structure looks transparent. Moreover, the representation should be such so that the gauge structure of the Yang-Mills theory appears natural, and that, for example, the known perturbative results can be readily recovered following the standard procedure. (In the one-matrix model of [2], the gauge transformations are essentially similarity transformations; therefore any $\infty \times \infty$ matrix can be naturally represented by its eigenvalues. The representation proposed

below similarly follows the general form of the $SU(\infty)$ Yang-Mills gauge transformations).

In order to capture the basic matrix properties of the gauge connections $A_i(\vec{x})$, as well as the corresponding non-abelian features of the planar theory, I start from the following correspondence relation

$$A_i(\vec{x}) \rightarrow a_i(\vec{x}, \alpha, \beta), \tag{1}$$

where α and β are two real parameters, and a_i's are functions of x, α, β. (In other words, the matrix indices, which play the role of internal parameters and which in the planar limit run from zero to infinity, are replaced by two continuous, external parameters α and β.) Given (1) how does one go about representing the non-abelian features of the theory? In particular, what is the form of a suitable representation of the commutator bracket? I propose the following identification

$$[A_i(\vec{x}), A_j(\vec{x})] \rightarrow \{a_i(\vec{x}, \alpha, \beta), a_j(\vec{x}, \alpha, \beta)\}, \tag{2}$$

where $\{,\}$ denotes the Poisson bracket with respect to α and β, or

$$\{a_i(\vec{x}, \alpha, \beta), a_j(\vec{x}, \alpha, \beta)\} = (\partial_\alpha a_i(\vec{x}, \alpha, \beta)\partial_\beta a_j(\vec{x}, \alpha, \beta) - \partial_\beta a_i(\vec{x}, \alpha, \beta)\partial_\alpha a_j(\vec{x}, \alpha, \beta)). \tag{3}$$

Thus all expressions containing the gauge connection and the commutator are to be replaced with the "identically" looking ones, after the translation defined in (1) and (2) has been applied. Also, the operation of tracing should be replaced by the operation of integration over the extra continuous parameters α and β

$$Tr \rightarrow \int d\alpha d\beta. \tag{4}$$

(The identical dictionary was suggested in a related context by the authors of [3], who made use of the equivalence between the $SU(\infty)$ Lie algebra and the algebra of area

preserving diffeomorphisms of a two-dimensional sphere S^2, parametrized by α and β. This equivalence is very important for the discussion of the ground state in what follows.)

The authors of [3] have also suggested the following translation which leads to a natural representation of the $SU(\infty)$ structure constants, namely

$$A_i^c(\vec{x})t^c \rightarrow \sum_{lm} a_i^{lm}(\vec{x})Y_{lm}(\alpha,\beta), \tag{5}$$

where t^c are the generators of $SU(\infty)$ and $Y_{lm}(\alpha,\beta)$ are the S^2 spherical harmonics. The $SU(\infty)$ structure constants are then identified with the structure constants of the area preserving diffeomorphisms of a two-sphere, defined in terms of the spherical harmonics basis [3]

$$\{Y_{lm}, Y_{l'm'}\} = f_{lm,l'm'}^{l''m''} Y_{l''m''}. \tag{6}$$

Then the expression defining the gauge transformations reads as follows

$$\delta a_i(\vec{x},\alpha,\beta) = \partial_i \Omega(\vec{x},\alpha,\beta) + g\{a_i, \Omega\}, \tag{7}$$

and the corresponding formula for the field strength is

$$F_{ij} = \partial_i a_j - \partial_j a_i + g\{a_i, a_j\}. \tag{8}$$

Given the above dictionary one can recover the standard results of pertubation theory, such as asymptotic freedom, starting from the Hamiltonian

$$H = \int d\vec{x} d\alpha d\beta \frac{1}{2}(p_i^2 + F_{ij}^2(a)), \tag{9}$$

where $p_i \rightarrow -i\frac{\delta}{\delta a_i}$ and the magnetic field F_{ij} is given by (8). (The well-known perturbative results can be easily recovered in the background-field approach.)

What is the nature of the ground state of the planar Yang-Mills theory in view of the correspondence relations (1) - (9)? (The ground state being the only surviving state

according to factorization.) In order to answer that question I wish to use the Hamiltonian version of the "constrained classical dynamics" formalism [4] (applicable both to vector and matrix planar field theories) which quite naturally incorporates the fundamental property of factorization through the following commutation relations between the planar gauge field and its conjugate momentum

$$[A_i(\vec{x}), P_j(\vec{y})] = i\delta_{ij}\delta(\vec{x}-\vec{y})|0\rangle\langle 0|. \tag{10}$$

(In other words, in the expansion of unity that appears in the usual canonical commutation relations $[A_i(\vec{x}), P_j(\vec{y})] = i\delta_{ij}\delta(\vec{x}-\vec{y})$

$$1 = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|, \tag{11}$$

only the first term, which is a projection operator, is kept. Note that (10) can be understood as $\langle 0|[A_i(\vec{x}), P_j(\vec{y})]|0\rangle = i\delta_{ij}\delta(\vec{x}-\vec{y})$, which in view of (1) implies $P_i(\vec{x}) \rightarrow p_i(\vec{x}, \alpha, \beta)$.) The dynamical equations of motion are given by the familiar expressions [4]

$$i[H_r, A_i(\vec{x})] = \dot{A}_i(\vec{x}), \tag{12}$$

and

$$i[H_r, P_i(\vec{x})] = \dot{P}_i(\vec{x}). \tag{13}$$

It is important to note that H_r represents the reduced Hamiltonian (reduced onto the ground state of the theory). Equations (10), (12) and (13) define the effective Hamiltonian version of the planar Yang-Mills theory.

Now I wish to consider the following concrete realization of such generalized quantum Hamiltonian dynamics based on a very particular representation of the projection operator in the planar commutation relations (10):

$$|0\rangle\langle 0| = \psi^\dagger \psi, \tag{14}$$

where $\psi^2 = \psi^{\dagger 2} = 0$, $\psi\psi^\dagger + \psi^\dagger\psi = 1$, i.e. ψ and $\psi^\dagger$ are fermionic operators. This representation is suggestive of a fermionic ground state.

That fact can be seen from the planar commutation relations (10), given the fermionic realization of the projector (14). Note, that due to the fact that $\psi^\dagger\psi$ is a fermion number operator, each phase cell $\Delta a_i \Delta p_i$ contains a single fermion. (The same fermion number operator serves as a generator of the area preserving diffeomorphisms of S^2, which is compatible with (1) and (2), so the ground state satisfies the Gauss law, that is, it is invariant under (7).) Given that, I conclude that the ground state is basically characterized by a certain region of the functional phase space $Da_i Dp_i$ which is characterized by the fundamental property of incompressibility according to the Liouville theorem. (These a_i and p_i configurations saturate the planar limit.) In other words, the following constraint (which is compatible with the Gauss law) is imposed on the functional phase space volume

$$\int Da_i Dp_i \theta(e - H) = 1, \tag{15}$$

where H denotes the Hamiltonian (9), e stands for the Fermi energy and θ is the usual step function.

Equation (15) tells us that the volume of the functional phase space fluid is to be normalized to one in such a way, as if there existed a single fermion placed at each phace space cell, and consequently, taking into account the Pauli exclusion principle, as if there existed, in the limit of a large number of cells, an incompressible fermionic fluid, with the Fermi energy e. By recalling that each phace space cell has a natural volume of the order of the Planck constant and that the planar limit corresponds to a situation where the number of cells goes to infinity, the product of the Planck constant and the number of cells can be adjusted to one (the reason being that the $1/N$ expansion formally corresponds to a "semiclassical" expansion, $1/N$ acting as an effective "Planck constant"). Hence follows the

relation (15), describing an incompressible drop of functional phase space of unit volume. (The appearance of fermions could be intuitively understood from the point of view of 't Hooft's double-line representation for the planar graphs [1]. The fact that such lines do not cross in the planar limit is achieved by attaching fermions to each line and using the exclusion principle.)

¿From this vantage point relation (15) gives a rather natural, even though implicit realization of the ground state of the planar Yang-Mills theory, that is compatible with the Gauss law.

By formally integrating over $p_i(\vec{x}, \alpha, \beta)$ in (15), a constraint imposed on the part of the configuration space variables $a_i(\vec{x}, \alpha, \beta)$ relevant for the planar limit, is obtained

$$\int Da_i \rho(a) = 1, \tag{16}$$

where the functional $\rho(a)$ is, again formally, defined by

$$\rho(a) = \int Dp_i \theta(e - H). \tag{17}$$

In other words the functional $\rho(a)$ corresponds to the volume of the functional momentum space that is relevant for the description of the ground state. Equation (16) provides another suitable representation of the ground state of the planar Yang-Mills theory, again compatible with the Gauss law. One could interpret (16) as

$$\int Da_i \delta(a_i - A_i) = 1, \tag{18}$$

which contains the same information as the starting equation (1).

Now it might seem reasonable to introduce the weight functional $\rho(a)$ as the appropriate new variable for an effective Hamiltonian description of the planar Yang-Mills theory. In order to accomplish this one might adopt, for example, the well-known collective-field

theory (or what is more appropriate in this case - collective-functional theory) approach of Jevicki and Sakita [5].

Here I would like to use the already established fermionic-fluid picture of the ground state (15). Then the effective planar Hamiltonian is given by

$$H_r = \int Da_i \int Dp_i (\int d^3x d\alpha d\beta \frac{1}{2}(p_i^2 + F_{ij}(a)^2))\theta(e - \int d^3x d\alpha d\beta \frac{1}{2}(p_i^2 + F_{ij}(a)^2)), \quad (19)$$

or in terms of a fermionic functional $\Psi(a)$ which describes the fermionic nature of the vacuum

$$H_r = \int d^3x d\alpha d\beta \int Da_i (\frac{1}{2}\frac{\delta\Psi^\dagger}{\delta a_i}\frac{\delta\Psi}{\delta a_i} + (\frac{1}{2}F_{ij}^2(a) - e)\Psi^\dagger(a)\Psi(a)), \quad (20)$$

where $\Psi^\dagger(a)\Psi(a)) = \rho(a)$, $\rho(a)$ being defined by (17). This formula can be understood as the usual expression for the ground state energy, written in a second quantized manner, after taking into account the fact that the ground state of the planar theory is fermionic, as implied by (15). (Given H_r it seems very difficult to recover any perturbative results precisely because now the true, non-perturbative vacuum is known, being described by (15).)

Note that the above expressions for the effective Hamiltonian contain functional integrals, the fact which tells us that we are not dealing with an ordinary field theory. (Here lies the fundamental difference between $N = \infty$ vector and matrix field-theory models: The planar limit of vector field-theory models is described in terms of suitable (bilocal) field variables, which implement the summation over all "bubble" diagrams, while the planar limit of matrix field-theory models, such as Yang-Mills theory, is described in terms of suitable functional variables, such as $\rho(a)$, or $\Psi(a)$, which implement the summation over all planar diagrams. In fact, the above formulation of the planar Yang-Mills theory has a certain flavor of what one might call "string field theory" or "non-local functional-field theory".)

One could use the weight functional $\rho(a)$ as a collective functional in the spirit of [5]. Then the expression (19) could be interpreted as the effective potential of the Jevicki-Sakita collective-functional Hamiltonian, after the application of the definition (17). The minimum of the effective potential, which determines the ground state of the planar Yang-Mills theory, is in turn given by (15). (The Fermi energy e plays the role of a Lagrange multiplier, imposing the constraint (16) in this approach.) Unfortunately, unlike in the one-matrix model case [2], a simple explicit expression for the collective functional $\rho(a)$ cannot be readily obtained, precisely because of the functional integral over the first term p_i^2 in the expression (19) for the effective collective functional Hamiltonian. The same term governs the physics of elementary excitations around the ground state (15). In other words, when expanded to second order in $\rho(a)$ around the minimum defined by (15), the same term determines the frequencies of small oscillations around the minimum of the effective potential (this meshes nicely with the intuitive picture developed in [6]). The fundamental frequency ω_0 is controlled by the density of states

$$\sigma(e) = \int Da_i Dp_i \delta(e - H), \tag{21}$$

as follows

$$\omega_0^{-1} = \sigma(e). \tag{22}$$

The fundamental frequency ω_0 is, at least formally, positive definite. (Again, an explicit functional expression in terms of $a_i(\vec{x}, \alpha, \beta)$ is not readily available).

In conclusion, let me summarize the essential points of the above realization of the planar Yang-Mills theory:

i) A suitable representation of the $SU(\infty)$ gauge connection is provided through (1) (along with the related prescriptions for the commutators, traces and $SU(\infty)$ structure

constants; the respective equations (2), (4) and (6)). This representation utilizes the equivalence between the $SU(\infty)$ Lie algebra and the algebra of area preserving diffeomorphisms of a two-sphere [3].

ii) A suitable representation of the ground state of the planar Yang-Mills theory, that is compatible with the Gauss law, is provided through (15). This representation is based on (i) and leads to a fermionic-fluid picture of the ground state.

iii) Given the fermionic-fluid picture of the ground state of the planar Yang-Mills theory, the effective planar Yang-Mills Hamiltonian is obtained; equations (19) and (20).

I thank V. P. Nair for crucial discussions and a critical reading of the original version of this article. I am grateful to Shyamoli Chaudhuri, Joseph Polchinski and Branko Urošević for comments. I also thank Bunji Sakita for drawing my attention to his work with Kavalov. I also wish to thank members of the University of Chicago Theory Group for their kind hospitality. This work was fully and generously supported by the Joy K. Rosenthal Foundation.

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